Imperial College London

MSc Quantum Fields and Fundamental Forces

Masters Thesis

A Review of Complex Geometry, and a Path to Generalised Calabi-Yau Manifolds

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1 Introduction

To the modern mind, geometry is often thought of as an offspring of arithemtic; yet to the ancient mind, in particular the Ancient Greeks, arithmetic was but a by-product of geometry. This flip in how we look at geometry, has probably come as a result of the modern developments of algebra and calculus, resulting in the study of geometry becoming the hybrid it is today. One such field that was concieved out of this mix is differential geometry. Complex geometry, which is a particular sect of differential geometry, is the focus of this thesis.

Complex geometry has been a subject of much interest in theoretical physics, mainly due to the folowing reason: solutions to certain supersymmetric string theories, are equivalent to the existence of a group of special complex manifolds [1]. That is, the geometry of complex manifolds, provides the link between supersymmetric string theories, and the local, observable world. To illustarte this point further, consider the background space of a 10 dimensional superstring theory, M_{10}. What sort of geometric structure must this space have? Locally, we know our geometry is Minkowski and thus somehow the space $\mathcal{M}_4$ must be involved. A natural way to proceed, is by postulating a compactification product space

$$M_{10} = \mathcal{M}_4 \times M_6$$

where the manifold M_6 is some 6-dimensional, compact manifold. In particualr, it must be sufficiently compactified so as to not be observable at low energies. The problem has been reduced to the finding of such a 6-dimensional manifold, with geometry that admits the Standard Model supersymmetry at low energy. The salient point is that the supersymmetry imposes a Calabi-Yau structure on the compactification manifold. That is, M_6 turns out to be a Calabi-Yau 3-fold which is a special type of 3-dimensional complex manifold (see chapter 9 for details).

In this thesis, I provide a review of complex geometry beginning with the complex manifold. The complex manifold is defined by two equivalent prescriptions; one using holomorphic maps, and the other by the introduction of an almost-complex structure. The almost-complex structure is examined in terms of an integrability condition, that determines whether the manifold is in fact complex or almost-complex. The chapter ends with an example of the canonical complex coordinates, which will characterise future complex manifold coordinate sysyems. Chapter 3 marks the first extensions of real structures to the complex manifold. These include the exterior deriavative, tensor field, and differential form field, which are complexified and decomposed into constituent holomorphic and antiholomorphic parts. They will form the basis for more advanced constructions later on such as the Ricci form and Kähler form. Chapters 4 and 5 explore the geometry resulting from the addition of a special Riemannian metric on an almost complex manifold. We outline the conditions of compatibility between the metric and almost complex structure, that will impose a special kind of geometry; namely the Kähler geometry. This geometry is also characterised by a connection that preserves holomorphicity. We find that this connection that develops naturally from Kähler geometry, actually coincides with the Levi-Civita connection. The remarkable property of Kähler geometric structures, is that they can all be determined (locally) if given the Kähler potential function. Once the Kähler manifold is defined, Hodge theory and Hodge decomposition theorems are extended to Kähler manifolds. Hodge decompositions lead naturally to a discussion of cohomology. Specifically, we consider the Dolbeault generalisation to the de Rahm cohomology. The resulting Hodge numbers that characterise the cohomology classes are discussed in detail. We then turn to a gauge-theory-perspective introduction of the Chern classes of complex vector bundles. These are characteristic classes and thus are subsets of the cohomology classes. It turns out, the first Chern class is essentially given by the Ricci form.

Finally, we get to constructing the Calabi-Yau manifold in chapter 9. We spend considerable time describing the various geometric aspects of this manifold with regards to supersymmetry, holonomy, Ricci flatness; making sure to highlight their interconnections. The condition for the existence of a Calabi-Yau metric is encapsulated by the Monge-Ampére equation, which we motivate by two different methods. We close the chapter by looking at further relations betweem the Hodge numbers and in particular write the Hodge diamond of the Calabi-Yau 3-fold. The final chapter represents a much briefer story on the generalised Calabi-Yau manifold that is used for compactifications with flux. It is outside the scope of this thesis to examine the generalised Calabi-Yau manifold to the same rigor as the Calabi-Yau manifold was discussed. Following the prescription [2], we quote the corresponding generlaisations to the mathematucal structures that were developed in chapter 2-5, giving references to more detailed accounts.

This thesis is aimed at graduate students of theoretical physics. I presuppose a standard graduate level knowledge of differential geometry, Riemannian geometry, and basic fibre bundles such as the tangent bundle (see [3] for an in depth examination of fibre bundles). I include an appendix on complex vector bundles and holomorphic bundles, as these are not typically included in an introductory course.

2 Complex Manifolds and Almost Complex Manifolds

2.1 Construction using Holomorphic Maps

A Holomorphic map $\mathbb{C}^m \to \mathbb{C}^n$ is a collection of complex valued functions $(f^1, ..., f^n)$ where f^i is holomorphic, meaning both f_1 and f_2 in $f = f_1 + if_2$ satisfy the Cauchy Riemann relations for each $z^\mu = x^\mu + iy^\mu$ where $\mu = \{1, ..., m\}$:

$$\frac{\partial f_1}{\partial x^\mu} = \frac{\partial f_2}{\partial y^\mu} \tag{1}$$
$$\frac{\partial f_2}{\partial x^\mu} = -\frac{\partial f_1}{\partial y^\mu}. \tag{2}$$

A **complex manifold** M, requries both a topological condition and a differentiability condition. A complex manifold must be a $dim_{\mathbb{R}} M = 2m$ topological space that is locally homeomorphic to $\mathbb{C}^m$, together with a smooth holomorphic differentiable structure. Explicitly the following must hold,

1. M is a topological space $(M, \mathcal{O})$
2. There exists an atlas $\{(U_i, \phi_i)\}$ of charts where $\{U_i\} \subset \mathcal{O}$ form an open cover on M and ϕ_i is a homeomorphism from U_i to an open set in $\mathbb{C}^m$.
3. Coordinate transition maps on manifold regions $U_i \cap U_j \neq \emptyset$ given by $\phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ are holomorphic maps, as defined above. In other words they depend only on z^μ and not on $\bar{z}^\mu$. An atlas consisting only of such transition functions is then labeled a holomorphic atlas. This procedure is independent of the choice of chart, so as to be a coordinate independent condition.

The complex structure of such a manifold is defined as the maximal holomorphic atlas. The union of two distinct holomorphic atlases may produce another complex structure, provided the axioms still hold, implying a complex manifold may in general have many complex structures.

By virtue of the above definition, a complex manifold of $dim_{\mathbb{C}} M = m$ (denoted a complex m-fold) may be identified with a real manifold of $dim_{\mathbb{R}} M = 2m$ (denoted a real $2m$-fold) because of the isomorphism $\mathbb{C}^m \cong \mathbb{R}^{2m}$. However the converse is not always true. Under what circumstances is a real manifold then also a complex manifold? In order to answer this question, we formulate an equivalent definition of a complex manifold, by introducing the **almost-complex structure**. We also find that a slighly weaker condition on the almost-complex structure defines an almost-complex manifold.

2.2 Construction using Almost Complex Structure

Definition 3.1: Let M be a real $2m$-fold. The almost-complex structure $J \in \Gamma(TM \otimes T^*M)$ is smooth $(1,1)$ tensor field, such that $J^i{}_l J^l{}_j = -\delta^i{}_j$. The manifold together with the almost-complex structure defines an almost-complex $2m$-fold, given as (M, J).

As a linear map, $J \in \Gamma(End(TM)) : TM \longrightarrow TM$. Taking a vector field $X \in \Gamma(TM)$, we define the action of the field J as $(JX)^i = J^i{}_j X^j$. Acting once more we have $J(JX)^i = J^l{}_i J^i{}_j X^j = -\delta^l{}_j X^j = -X^j$, noting we used the almost-complex property. This shows that J^2 acts as minus the identity operator, and therefore, is the manifold generlisation of multiplication by $\pm\imath$. In other words it gives $\Gamma(TM)$ a complex vector space structure. This prescription is always valid locally [4]; in other words, any $2m$-dimensional manifold admits an almost-complex structure J where

$$J^2 = -I_{2m} \tag{3}$$

Given a basis $\{\frac{\partial}{\partial x^i}\}$ and dual basis $\{dx^j\}$, the almost-complex structure can then be written as

$$J = J^i{}_j \frac{\partial}{\partial x^i} \otimes dx^j \tag{4}$$

where the components $J^i{}_j$ satisfy the equation in the above definition.

In the case of a complex manifold however, this tensor field can also be patched globally on the manifold. To ellucidate this point further, we introduce the Nijenhuis tensor field $N \in \Gamma(TM \otimes T^*M \otimes T^*M)$, defined by its action $N : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$ given by

$$N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] \tag{5}$$

where [,] is the usual Lie Bracket operation on vector fields. In component form we have $N^i{}_{jk} = J^l{}_j(\partial_l J^i{}_k - \partial_k J^i{}_l) - J^l{}_k(\partial_l J^i{}_j - \partial_j J^i{}_l)$. We then state two theorems to complete the connection between almost-complex structure and complex manifolds

Theorem 3.1: An almost-complex structure J is integrable $\Longleftrightarrow N(X,Y) = 0 \ \forall \ X,Y \in \Gamma(TM)$

Theorem 3.2 (Newlander and Nirenberg Theorem [6]): Let (M, J) be an almost-complex 2m-fold. The almost-complex structure J is integrable $\Longleftrightarrow$ The almost-complex manifold is actually a complex manifold.

These theorems imply that integrability is both nessessary and sufficient, in order to establish a global covering of complex coordiantes on a manifold. If J is indeed integrable, the almost-complex structure is labeled the complex structure, and the almost-complex manifold becomes a complex manifold, denoted (M,J). The interrelation between the two definitions of a complex manifold can be summarised by the following:

Holomorphic atlas $\Longleftrightarrow$ Integrable almost-complex structure $\Longleftrightarrow$ Complex manifold

We now complexify the notions of a tangent bundle and the corresponding complex structure endomorphism J:

$$TM \longrightarrow T_{\mathbb{C}}M = TM \otimes \mathbb{C} \tag{6}$$

$$J \longrightarrow J_{\mathbb{C}} : TM \otimes \mathbb{C} \longrightarrow TM \otimes \mathbb{C} \tag{7}$$

where for conveniance we drop the $\mathbb{C}$ from the complex structure. But what are the geometrical implications of these complexifications? The eigenvalues of the complex structure are $\pm\imath$, implying a natural projection of the tangent bundle into two isomorphic subspaces.

$$T_{\mathbb{C}}M = TM \otimes \mathbb{C} = (T^{(1,0)}M) \otimes (T^{(0,1)}M) \tag{8}$$

where $(T^{(1,0)}M)$ is conjugate to $(T^{(0,1)}M)$, and they are respectfully labeled the holomorphic and anti-holomorphic tangent bundle's. Correspondingly, the cotangent bundle can also be complexified as $T_{\mathbb{C}}{}^*M = T^*M \otimes \mathbb{C} = (T^{*(1,0)}M) \otimes (T^{*(0,1)}M)$.

To make this more concrete we look at the canonical coordinate example. Consider a real (co)tangent space at a point p given by $(T_p{}^*M)$ T_pM, which we complexify to $(T^{\mathbb{C}^*}{}_pM)$ $T^{\mathbb{C}}{}_pM$ by the following prescription. A complex (co)vector $Z \in (T^{\mathbb{C}^*}{}_pM)T^{\mathbb{C}}{}_pM = X + \imath Y$ where $X, Y \in (T_p{}^*M)T_pM$. Define a complex (dual) basis for the complexified (co)tangent space out of the real basis we introduced in the previous section:

$$
\begin{aligned}
\frac{\partial}{\partial z^{\mu}} =& \frac{1}{2}\{\frac{\partial}{\partial x^{\mu}} - \imath\frac{\partial}{\partial y^{\mu}}\} && (9)\\
\frac{\partial}{\partial \bar{z}^{\mu}} =& \frac{1}{2}\{\frac{\partial}{\partial x^{\mu}} + \imath\frac{\partial}{\partial y^{\mu}}\} && (10)\\
dz^{\mu} =& dx + \imath dy && (11)\\
d\bar{z}^{\mu} =& dx - \imath dy && (12)
\end{aligned}
$$

Note the relabelling of the (co)tangent vectors from $(\{dx^{\mu}\})$ $\{\frac{\partial}{\partial x^{\mu}}\}$ to $(\{dy^{\mu}\})$ $\{\frac{\partial}{\partial y^{\mu}}\}$ for $\mu = \{m+1, ..., m\}$. As in the real case these satisfy the same duality conditions, namely the only non vanishing products are $< dz^{\mu}, \frac{\partial}{\partial z^{\nu}} >=< d\bar{z}^{\mu}, \frac{\partial}{\partial \bar{z}^{\nu}} >= \delta^{\mu}{}_{\nu}$.

The complex structure then acts as $J_p(X + \imath Y) = J_p X + \imath J_p Y$ for $X, Y \in T_pM$. We define its action on the holomorphic and anti-holomorphic basis vectors as

$$
\begin{aligned}
J_p(\frac{\partial}{\partial z^{\mu}}) =& \imath(\frac{\partial}{\partial z^{\mu}}) && (13)\\
J_p(\frac{\partial}{\partial \bar{z}^{\mu}}) =& -\imath(\frac{\partial}{\partial \bar{z}^{\mu}}), && (14)
\end{aligned}
$$

noting that the complex structure property is accordingly satisfied.

A tensorial expression for J using the complex bases above is then $J_p = \imath dz^{\mu} \otimes \frac{\partial}{\partial z^{\mu}} - \imath d\bar{z}^{\mu} \otimes \frac{\partial}{\partial \bar{z}^{\mu}}$ with components given by

$$
J_p = \begin{bmatrix} iI_m & 0 \\ 0 & -iI_m \end{bmatrix} \tag{15}
$$

The components of the complex structure are all constant. Looking at the components of the Nijenhuis tensor, we see they must vanish due to the partial derivatives in each term. This implies integrability of the almost-complex structure. By Theorem 3.2, this confirms these complex coordinates can be used globally, and the manifold is not just almost-complex but complex.

Using the almost-complex structure, the complexified tangent space is split into disjoint vector spaces $T_pM^{\mathbb{C}} = {T_p}^{1,0}M \oplus {T_p}^{0,1}M$ where

$$
\begin{aligned}
{T_p}^{1,0}M =& \{Z \in T_pM^{\mathbb{C}} : J_pZ = +iZ\} \qquad (16) \\
{T_p}^{0,1}M =& \{Z \in T_pM^{\mathbb{C}} : J_pZ = -iZ\}. \qquad (17)
\end{aligned}
$$

For notational convenience we define these spaces ${T_p}^{1,0}M := {T_p}^{+}M$ and ${T_p}^{0,1}M := {T_p}^{-}M$. The (anti)holomorphic tangent bundles (see appendix for an introduction to holomorphic vector bundles) are then given by

$$
TM^{\pm} := \bigcup_{p \in M} {T_p}^{\pm}M. \qquad (18)
$$

Projection operators may then be used to project onto these subspaces as $P^{\pm} : T_pM^{\mathbb{C}} \to T_pM^{\pm}$. We define them as $P^{\pm} = \frac{1}{2}(I_{2m} \mp \imath J)$. The eigenvalue equation $J_pP^{\pm}Z = \pm iP^{\pm}Z$ then shows that $Z^{\pm} = P^{\pm}Z \in T_pM^{\pm}$. Finally a general complex vector $Z \in T_pM^{\mathbb{C}}$ is decomposed as

$$
Z = Z^{+} + Z^{-} \qquad (19)
$$

Naturally, a holomorphic vector is a vector $Z_1 \in T_pM^{+}$, with a coordinate basis $\frac{\partial}{\partial z^{\mu}}$. An anti-holomorphic vector is a vector $Z_2 \in T_pM^{-}$ with coordiante basis $\frac{\partial}{\partial \bar{z}^{\mu}}$.

3 Tensors and Exterior Forms on Complex Manifolds

3.1 Tensor Field Complexification and Decomposition

Motivated by the decomposition of vectors, we may also decompose a tensor field into holomorphic and anti-holomorphic parts. Superscript indices decompose as $S^a = S^\alpha + S^{\bar{\alpha}}$ and subscript indices decompose as $T_b = T_\beta + T_{\bar{\beta}}$. The unbarred indices form tensor products of holomorphic (co)tangent spaces and the barred indices the anti-holomorphic (co)tangent spaces. For example, if we consider complexifying the (co)tangent bundles on a real manifold to give complex valued tensor fields; a (p, q)-tensor field would be an element of $\Gamma(\otimes^p T_\mathbb{C} M \otimes^q {T_\mathbb{C}}^* M)$, where each complexified bundle has the unique decomposition given previously.

3.2 Differential Form Field Complexification and Decomposition

The next question one should ask is how to complexify a real differential form, and what do forms look like on complex manifolds (note these are different questions)? A real differential form may be complexified with the addition of an imaginary form of the same order. Let $\omega, \eta \in {\Omega_p}^q(M)$. We define a unique complex differential q-form at p as $\zeta = \omega + \imath\eta$. More formally, a complex q-form is a smooth section of $\wedge^q {T_\mathbb{C}}^* M$ and the vector space of complex q-forms is $\Gamma(\wedge^q {T_\mathbb{C}}^* M) := {\Omega^q}_\mathbb{C}$. However, we can use the complex cotangent decomposition to state the following theorem

Theorem 3.2.1:

$$\wedge^q T_{\mathbb{C}}{}^* M = \bigoplus_{r+s=q} \wedge^{r,s} M \tag{20}$$

where $\wedge^{r,s} M := \wedge^r T^{*(1,0)} M \otimes \wedge^s T^{*(0,1)} M$. A section of this defines a bi-degree (r, s)-form.

Let us motivate this in a slightly less abstarct way. Let M be a complex manifold M with $T_p M^{\mathbb{C}}$ and $\Omega_p{}^q(M)^{\mathbb{C}}$. A bi-degree (r, s) is an $r + s = q$ form ω, for which $\omega(V_1, ...V_q) = 0$ unless r of the $V_i \in T_p{}^{(1,0)}(M)$, and s of the $V_i \in T_p{}^{(0,1)}(M)$. For example, dz^μ is of bidegree $(1, 0)$, $d\bar{z}^\mu$ is of bidegree $(0, 1)$. An (r, s) form thus has a basis $dz^{\alpha_1} \wedge ... \wedge dz^{\alpha_r} \wedge d\bar{z}^{\beta_1} \wedge ...d\bar{z}^{\beta_s}$ at each point $p \in M$. In general a form of bidegree (r, s) labeled $\omega^{(r,s)}$, is given by

$$\omega^{(r,s)} = \frac{1}{r!s!} \omega_{\alpha_1 ... \alpha_r \beta_1 ... \beta_s} dz^{\alpha_1} \wedge ... \wedge dz^{\alpha_r} \wedge d\bar{z}^{\beta_1} \wedge ...d\bar{z}^{\beta_s} \tag{21}$$

where the antisymmetric components are defined in the usual way for tensors. Note that in order for this structure to be useful it must be chart independent, thus a bi-degree (a, b) will remain a bi-degree (a, b) in a different coordinate system. Theorem 3.2.1 then implies that a general complex q-form ω, is uniquely decomposed into a sum of its bi-degrees as

$$\omega = \sum_{r+s=q} \omega^{(r,s)}. \tag{22}$$

In terms of vector spaces the decomposition becomes

$$\Omega^q(M)^{\mathbb{C}} = \bigoplus_{r+s=q} \Omega^{r,s}(M) \tag{23}$$

where $\Omega^{r,s}(M)$ is the space of (r, s) bi-degrees.

3.3 Dolbeault Operators

Due to (23), the exterior derivative of a complex q-form may be defined in terms of the exterior derivative of the component (r, s) forms where $r + s = q$. As in the non-complex case the exterior deriavtive increases the q-form form to a (q+1)-form, or equivalently to a collection of unique (r+s+1)-form terms. However a (r+s+1)-form may be split into two different bidegrees; a bidgree (r+1, s)-form and a (r, s+1)-form. Therefore it is natural to break the exterior derivative into the sum of two linear operators acting on each part of a the bidegree seperately. This motivates the following definition.

Definition 3.3.1:

The Dolbeault operators ∂ and $\bar{\partial}$ form the sum $d = \partial + \bar{\partial}$, where $\partial : \Omega^{r,s}(M) \to \Omega^{r+1,s}(M)$, $\bar{\partial} : \Omega^{r,s}(M) \to \Omega^{r,s+1}(M)$.

The action on a general $q = r + s$ form is then

$$\partial\omega = \sum_{r+s=q} \partial\omega^{(r,s)} \tag{24}$$

$$\bar{\partial}\omega = \sum_{r+s=q} \bar{\partial}\omega^{(r,s)}. \tag{25}$$

For example, consider a (1, 1) form $\omega = \omega_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu$. $\bar{\partial}\omega = \frac{\partial\omega_{\mu\bar{\nu}}}{\partial\bar{z}^\sigma} d\bar{z}^\sigma \wedge dz^\mu \wedge d\bar{z}^\nu$. Note that the nilpotency of the external deriavtive implies the Dolbeault operators obey the identites $\partial\partial = \bar{\partial}\bar{\partial} = \partial\bar{\partial} + \bar{\partial}\partial = 0$.

Definition 3.3.2:

Let M be a complex manifold and $\omega \in \Omega^{r,0}(M)$ satisfy $\bar{\partial}\omega = 0$. That is ω is an (r, 0) form that is $\bar{\partial}$ closed. ω is then called a **holomorphic r-form**.

On a chart (U, ϕ) such an r-form can be expressed as $\omega = \frac{1}{r!}\omega_{\mu_1...\mu_r} dz^{\mu_1} \wedge ... \wedge dz^{\mu_r}$ where

$$\frac{\partial}{\partial \bar{z}^\lambda}\omega_{\mu_1...\mu_r} = 0. \tag{26}$$

This is simply the holomorphic condition (on the local chart) for each function $\omega_{\mu_1...\mu_r}$. The canonical vector bundle [7] (see appendix) of a complex manifold is defined as $K_M = \wedge^{m,0}M$. Sections of this bundle are the (m, 0) holomorphic forms.

We now turn to introducing various complex vector spaces that will be needed in later chapters. Consider the $\bar{\partial}$-closed and $\bar{\partial}$-exact (r, s) forms. That is, the $\omega \in \Omega^{r,s}(M)$ that satisfy $\bar{\partial}\omega = 0$, and the $\omega = \bar{\partial}\omega_1$ for $\omega_1 \in \Omega^{(r,s-1)}$.

Definition 3.3.3: We respectfully define the set of $\bar{\partial}$-closed (r, s) forms as the **(r, s)-cocycle** $= Ker(\bar{\partial} : \Omega^{r,s}(M) \longrightarrow \Omega^{r,s+1}(M)) := Z_{\bar{\partial}}{}^{r,s}(M)$, and the set of $\bar{\partial}$-exact (r, s) forms as the **(r, s)-coboundary** $= im(\bar{\partial} : \Omega^{r,s-1}(M) \longrightarrow \Omega^{r,s}(M)) := B_{\bar{\partial}}{}^{r,s}$.

Clearly the holomorphic r-forms are a subset of the (r, s)-cocycle, and $B_{\bar{\partial}}{}^{r,s}(M) \subset Z_{\bar{\partial}}{}^{r,s}$ as every $\bar{\partial}$-exact form must be $\bar{\partial}$-closed due to the nilpotency of $\bar{\partial}$.

There are two fundamental differential forms that must be considered on the path to Calabi-Yau geometry; these are the Kähler and Ricci forms. However in order to define these, we must add some more structure to the complex manifold. Specifically, we introduce a Riemannian metric and impose a condition on it to define a Hermition metric. Unless stated otherwise, from now on we work exclusively with (atleast) an almost-complex manifold structure of $dim_{\mathbb{C}} = m$ which we denote by (M, J).

4 Hermition Geometry

4.1 Hermition Metric and Hermition Manifold

Definition 4.1.1:

On an almost-complex manifold equipped with a Riemannian metric g, the metric is defined **Hermition** if it satisifes

$$g_p(J_pX, J_pY) = g_p(X, Y) \tag{27}$$

$\forall\ p \in M$ and $X, Y \in T_pM$.

We say that the metric is compatible with J. Note infact that J_pX is perpendicular to X with repsect to the Hermition metric. The components are then $g_{ij} = J^l{}_i J^k{}_j g_{lk}$. The manifold is then called Hermition and is represented by (M, J, g)

Theorem 4.1.1:

A Hermition metric can always be found on a complex manifold.

Proof:

Consider a new metric $\hat{g}_p(X, Y) = \frac{1}{2}[g_p(X, Y) + g_p(J_pX, J_pY)]$. The Hermition condition is satisfied due to the cancellation of the minus signs from each $J^2 = -I$. It is also automatically positive definite. □

We now extend the metric tensor domain to include complex vector imputs $Z = X + \imath Y$, $W = U + \imath V \in T^{\mathbb{C}}{}_pM$:

$$g_p(Z, W) = g_p(X, U) - g_p(Y, V) + \imath[g_p(X, Y) + g_p(Y, U)]. \tag{28}$$

The metric is then a complex bi-linear, symmetric, positive definite map $T_pM^{\mathbb{C}} \otimes T_pM^{\mathbb{C}} \rightarrow \mathbb{C}$. Using the canonical coordinates for the holomorphic and anti-holomorphic basis vectors, the two unmixed metric components become $g_{\mu\nu} = g(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu})$, $g_{\bar{\mu}\bar{\nu}} = g(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu})$, and similarly for the two mixed components. Symmetries follow naturally: $g_{\mu\nu} = g_{\nu\mu}$, $g_{\bar{\mu}\nu} = g_{\nu\bar{\mu}}$, $g_{\bar{\mu}\bar{\nu}} = g_{\bar{\nu}\bar{\mu}}$, $\bar{g}_{\mu\bar{\nu}} = g_{\bar{\mu}\nu}$, $\bar{g}_{\mu\nu} = g_{\bar{\nu}\bar{\mu}}$.

In complex components, (27) constrains the unmixed indices to vanish $g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} = 0$, leaving block diagonal metric components $g_{ab} = g_{\alpha\bar{\beta}} + g_{\bar{\alpha}\beta}$. In tensor notation it can be written as

$$g = g_{\mu\bar{\nu}} dz^\mu \otimes d\bar{z}^\nu + g_{\bar{\mu}\nu} d\bar{z}^\mu \otimes dz^\nu, \tag{29}$$

where we can now more clearly see the Hermition metric is a map $T^{(1,0)}M \otimes T^{(0,1)}M \longrightarrow \mathbb{C}$. Consider the following inner product on $T^{(1,0)}M$:

$$h(X, Y) = g(X, \bar{Y}) for X, Y \in T^{(1,0)}M \tag{30}$$

It is easily shown that h is both positive definite and Hermition. Further note that $h(\bar{X, Y}) = h(Y, X)$, indicating why we use the familiar Hermition label for the above metric.

4.2 Connections, Tensors and Forms on Hermition Manifolds

In order to define a covariant derivative on a Hermition manifold, we must first define a connection compatiblle with the complex structure. Only then, can we establish connection components on the holomorphic and anti-holonorphic (co)tangent complexified bundles. We first assume that on a Hermition manifold, a holomorphic vector remains holomorphic after parralel transport. Parallel transport a holomorphic vector X, an infinitesimal coordiante displacement (select a chart) dz^μ, from point $p \to q$, giving $X \to X'$. The components of the parallely transported vector differ from the old by $-X^\sigma \Gamma^\lambda_{\mu\sigma} dz^\mu$, which defines the connection components on the holomorphic basis vector. We summarise below all non vanishing connection components:

$$
\begin{aligned}
\nabla_\mu \frac{\partial}{\partial z^\nu} &= \Gamma^\lambda_{\mu\nu}(z) \frac{\partial}{\partial z^\lambda} \\
\nabla_{\bar\mu} \frac{\partial}{\partial \bar z^\nu} &= \Gamma^{\bar\lambda}_{\bar\mu\bar\nu}(z) \frac{\partial}{\partial \bar z^\lambda} \\
\nabla_\mu dz^\nu &= -\Gamma^\nu_{\mu\lambda}(z) dz^\lambda \\
\nabla_{\bar\mu} d\bar z^\nu &= -\Gamma^{\bar\nu}_{\bar\mu\bar\lambda}(z) d\bar z
\end{aligned}
$$

where $\Gamma^\nu_{\mu\lambda}$ and $\Gamma^{\bar\nu}_{\bar\mu\bar\lambda} = \bar\Gamma^\nu_{\mu\lambda}$ are the only non zero components of the connection coefficients.

Definition 4.2.1:

A Hermition connection is the unique connection on a Hermition manifold when the metric is covariantly conserved $\nabla_\kappa g_{\mu\bar\nu} = \nabla_{\bar\kappa} g_{\mu\bar\nu} = 0$, and the unmixed connection components are the only non vanishing connection components .

The metric connection imposes the components

$$\Gamma^\lambda_{\kappa\mu} = g^{\bar\nu\lambda} \partial_\kappa g_{\mu\bar\nu} \tag{31}$$

$$\Gamma^{\bar\lambda}_{\bar\kappa\bar\nu} = g^{\bar\lambda\mu} \partial_{\bar\kappa} g_{\mu\bar\nu}. \tag{32}$$

J is also covariantly constant with respect to the Hermition connection

$$(\nabla_\kappa J)_\nu{}^\mu = 0 \tag{33}$$

which one can verify by using the tensorial expression of J along with the action of the connection on basis (co)tangent vectors.

Using the Hermition connection, the non zero components of the Torsion tensor are

$$T^\lambda{}_{\mu\nu} = g^{\bar{\sigma}\lambda}(\partial_\mu g_{\nu\sigma} - \partial_\nu g_{\mu\sigma}) \tag{34}$$

$$T^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}} = g^{\bar{\lambda}\sigma}(\partial_{\bar{\mu}} g_{\bar{\nu}\sigma} - \partial_{\bar{\nu}} g_{\bar{\mu}\sigma}) \tag{35}$$

Similarly for the Riemann curvature tensor, the non-zero components are $R^\kappa{}_{\sigma\bar{\mu}\nu}, R^\kappa{}_{\sigma\mu\bar{\nu}}, R^{\bar{\kappa}}{}_{\bar{\sigma}\bar{\mu}\nu}, R^{\bar{\kappa}}{}_{\bar{\sigma}\mu\bar{\nu}}$. Due to the anti-symmetry in the final two indices, we are left with only two independent components

$$R^\kappa{}_{\sigma\bar{\mu}\nu} = \partial_{\bar{\mu}}(g^{\bar{\lambda}\kappa}\partial_\nu g_{\sigma\bar{\lambda}}) \tag{36}$$

$$R^{\bar{\kappa}}{}_{\bar{\sigma}\mu\bar{\nu}} = \partial_\mu(g^{\bar{\kappa}\lambda}\partial_{\bar{\nu}} g_{\lambda\bar{\sigma}}). \tag{37}$$

Contract the first two indices of (36) as $R^\kappa{}_{\kappa\mu\bar{\nu}} = -\partial_\nu(g^{\kappa\bar{\lambda}}\partial_\mu g_{\kappa\bar{\lambda}})$ (note we used the symmetry $R^\kappa{}_{\lambda\bar{\mu}\nu} = -R^\kappa{}_{\kappa\nu\bar{\mu}}$). Using the identity $\delta det(g_{\mu\bar{\nu}}) = det(g_{\mu\bar{\nu}})g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$, we rewrite the above contraction as $R^\kappa{}_{\kappa\mu\bar{\nu}} = -\partial_\mu\partial_{\bar{\nu}} \log det(g_{\mu\bar{\nu}})$. We now define the Ricci form $\Re$, which is a (1, 1) bi-degree (also a real 2-form) with complex components [8] $\Re_{\mu\bar{\nu}} = R^\kappa{}_{\kappa\mu\bar{\nu}}$ as

$$\Re := \imath\Re_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu = -\imath\partial_\mu\partial_{\bar{\nu}} \log det(g_{\alpha\bar{\beta}})dz^\mu \wedge d\bar{z}^\nu = \imath\partial\bar{\partial} \log det(g_{\alpha\bar{\beta}}) \tag{38}$$

If we consider the identity $\partial\bar{\partial} = -\frac{1}{2}d(\partial - \bar{\partial})$, the Ricci form is clearly closed (but not necessarily exact). We will see later this forms defines a special co-homoology class called the first Chern classes, and how it relates to the Calabi conjecture.

Consider now a (0, 2) tensor field Ω on a Hermition manifold. Its action on vectors $X, Y \in T_p(M)$ is defined as

$$\Omega_p(X, Y) = g_p(J_pX, Y) \tag{39}$$

Properties:

1. $\Omega(X, Y) = -\Omega(Y, X)$: Anti-symmetric
2. $\Omega(JX, JY) = \Omega(X, Y)$: Invariant under J action (Compatible with J)
3. If $\Omega(X, Y) = 0 \ \forall \ Y \in T_p(M)$, then $X = 0$: Non-degenerate
4. $\Omega_{ij} = \frac{1}{2}J^l{}_i g_{lj}$: Real components

An anti-symmetric (0, 2) tensor field can be made into a 2-form in the usual way. Complexifying the domain from $T_pM \to T_pM^{\mathbb{C}}$, the complex components of Ω are found as $\Omega_{\mu\nu} = \Omega_{\bar{\mu}\bar{\nu}} = 0$ and $\Omega_{\mu\bar{\nu}} = -\Omega_{\bar{\nu}\mu} = \imath g_{\mu\bar{\nu}}$. The resulting form is called the Kähler form. We then say that the existence of such a structure on a Hermition manifold ensures the compatibility of the tripple (g, J, Ω). Ω can then be written in differential form notation as

$$\Omega = -J_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu \tag{40}$$

where $\Omega_{\mu\bar{\nu}} = -J_{\mu\bar{\nu}} := \imath g_{\mu\bar{\nu}}$. Note Ω is a real form.

Theorem 4.2.1: All complex manifolds are orientable.

The proof essensially follows from the fact that the Kähler form on a Hermition manifold is a real, non degenerate 2-form. Its maximal wedge power is given by $\Omega_1 \wedge ... \wedge \Omega_m$, and is nowhere vanishing. Since any complex manifold can be made hermition, a natural volume form exists on any complex manifold, thus ensuring all complex manifolds are inherantly orientable.

5 Kähler Geometry

5.1 Conditions for Kählerity

Definition 5.1.1:

A **symplectic form** is a closed, non-degenerate, differential 2-form. If a symplectic form is assigned smoothly to a manifold, the manifold is said to be symplectic.

Now consider the Kähler form Ω defined on a hermition manifold. If we impose its closure, i.e the condition $d\Omega = 0$, it defines a symplectic form on a particular symplectic manifold. We define this special type of symplectic manifold: the Kähler manifold. We show later that in addition to the symplectic structure, a Kähler manifold comes with a compatible integrable complex structure. Note that a symplectic manifold automatically carries an almost-complex structure, but there is no apriori integrability imposed on it. If we were to relax this integrability condition, our Kähler manifold would generalise to an almost Kähler manifold.

Lets now unpack the geometric consequences. Explicitly, the closure of the Kähler form can be expressed as a condition on the partial derivatives of the metric:

$$(\partial + \bar{\partial})(-J_{\mu\bar{\nu}} dz^{\mu} \wedge d\bar{z}^{\nu}) = 0 \Longrightarrow \partial_{\lambda} g_{\mu\bar{\nu}} = \partial_{\mu} g_{\lambda\bar{\nu}} \quad (41)$$

$$\partial_{\bar{\lambda}} g_{\mu\bar{\nu}} = \partial_{\bar{\nu}} g_{\mu\bar{\lambda}}. \quad (42)$$

Equivalentlly the closure of the Kähler form is imposed by the covariant conservation of the almost-complex structure on the hermition manifold (which all complex manifolds satisfy)

Theorem 5.1.1 :

$$d\Omega = 0 \Longleftrightarrow \nabla_{\mu} J = 0[8] \quad (43)$$

where ∇ is now the familiar Levi-Civita connection. We say that the complex structure is parallel on a Kähler manifold. Unlike a hermition manifold which doesn't require intergability, a Kähler manifold does, and is thus always a subset of the complex manifolds.

Another important consequence of Kählerity is the existence of a Kähler potential K_i. In particular, we find that any Kähler metric can be expressed locally by this potential. This is a very powerful consequence of Kähler geometry, for once the Kähler metric is determined, all other geometric quantities can be calculated using the methods outlined already. This then reduces the geometry to the finding of a single scalar fucntion.

Theorem 5.1.2 :

Given a hermition metric on a patch U_i, the metric components can always be written as

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} K_i \tag{44}$$

.

where $K_i \in O(U_i)$, with $O(U_i)$ the space of holomorphic functions $f_i : U_i \longrightarrow \mathbb{C}$. Its partial derivatives clearly satisfy the Kähler condition. Given the above theorem, the Kähler form on a patch U_i becomes

$$\Omega = \imath \partial \bar{\partial} K_i. \tag{45}$$

Likewise, any closed (1, 1)-form can be expressed locally in this way. Similar to a Maxwell or Yang-Mills potential, the function can not generally be patched up globally on the maifold, as (anti)holomorphic transition functions will always vanish under the partial deriavtives. A Kähler potential is thus generally of the form $K_i = K_j + \phi^1(z) + \phi^2(\bar{z})$. In fact, for a compact Kähler manifold, there exists no globally defined K_i.

For the Kähler metric, the Riemann tensor acquires an extra symmetry such that the components of the Ricci form satisfy $\Re_{\mu\bar{\nu}} = R^{\kappa}_{\kappa\mu\bar{\nu}} = R^{\kappa}_{\mu\kappa\bar{\nu}} = Ric_{\mu\bar{\nu}}$. Hence why the name Ricci for the form previously defined was appropriate. Further note that combining (41, 42) with the torsion tensor components (34, 35) ensure that the torsion on a Kähler manifold is vanishing. This shows that the Kähler coincides with the Levi Civita connection. Torsionless geometry is also familiar from General Relativity.

We now consider the holonomy of a Kähler manifold. This will give us our final equivalent Kählerity condition. Due to the parallel complex structure, the Levi-Civita connection has no mixed components. Parallel transport then preserves holomorphicity which thus implies a constraint on the holonomy.

Consider parallely transporting a holomorphic vector $X|_p$ around a loop L to obtain $X'|_p$. The components (selecting a chart) $X'^{\mu} = X^{\mu} h_{\nu}{}^{\mu}$ where $h_{\nu}{}^{\mu}$ are the elements of the holonomy group. The critical point is that the connection preserves the length of the complex vector after parallel transport, meaning $h_{\nu}{}^{\mu}$ is an element of (or a sub-group of) $U(m) \subset O(2m)$.

We may now give a comprehensive definition of a Kähler manifold, incoorporating the various equivalent structures we have defined in this section (noting we have not proved all the equivalencies).

Definition 5.1.2 :

1. A Kähler manifold is a Hermition manifold with a closed Kähler form Ω.
2. A Kähler manifold is a symplectic manifold with symplectic form Ω, and a compatible, integrable complex structure.
3. A Kähler manifold has a holonomy group $U(m)$.

The metric of a Kähler manifold is then said to be a Kähler metric. Unlike hermition metrics however, not all complex manifolds admit Kähler metrics. Note additionally, that the line element of a Kähler metric is $ds^2 = 2g_{\alpha\bar{\beta}} dz^{\alpha} d\bar{z}^{\bar{\beta}}$.

A simple example of a Kähler manifold is a Riemann surface. This is a 1-dimesnional compact, orientable complex manifold. Infact, the bosonic world sheet in string theory happens to be a Riemann surface. Its symmetries are studied in [9].

6 Hodge Theory on Complex Manifolds

We start by generalising the Hodge star operation to complex manifolds endowed with Hermition / Kähler metrics, and complex bi-linear forms. Analogous to the Hodge star action on a real form, the hodge star action on a bi-linear form utilises the natural isomorphism ($\bar{*}$:)$\Omega^{r,s}(M) \longrightarrow \Omega^{m-r,m-s}(M)$. Note $*\beta =: \bar{*\beta} = *\bar{\beta}$. If instead we were to consider complexified (r+s=q)-forms, the Hodge star operation becomes an isomorphism of complex vector bundles, represented $* : \Lambda^q T^*{}_{\mathbb{C}}M \longrightarrow \Lambda^{2m-k} T^*{}_{\mathbb{C}}M$.

Let M be a compact Hermition manifold of $dim_{\mathbb{C}} M = m$. We define an L^2 inner product [7]: $<\alpha, \beta>= \int_M (\alpha, \beta)\Omega^m$, where $\alpha, \beta \in \Omega^{r,s}(M)$ and Ω^m is the volume form defined previously on a Hermition manifold. $<\alpha, \beta>$ is then a complex function, bilinear in α and $\bar{\beta}$. $*\beta$ is then defined as the unique $(2m-q)$-form such that $\alpha \wedge *\beta = (\alpha, \beta)\Omega^m \ \forall \ \alpha$.

We further define the adjoint dolbeault operators:

$$d^\dagger \beta = -*d*\beta \tag{46}$$

$$\partial^\dagger \beta = -*\partial*\beta : \Omega^{r,s}(M) \longrightarrow \Omega^{r-1,s}(M) \tag{47}$$

$$\bar{\partial}^\dagger \beta = -*\bar{\partial}*\beta : \Omega^{r,s}(M) \longrightarrow \Omega^{r,s-1}(M) \tag{48}$$

Noting that $\left(\partial^\dagger\right)^2 = \left(\bar{\partial}^\dagger\right)^2$, just as one would expect.

6.1 Laplacians, Harmonic Forms and Decomposition Theorems

In order to formulate Hodge Theory on a Hermition manifold, the analog of the Laplacian must also be defined.

The Laplacian on a real manifold $\Delta = (dd^\dagger + d^\dagger d)$, is naturally extended to a Hermition manifold by considering the Dolbeault operators seperately:

$$\Delta_\partial = (\partial + \partial^\dagger)^2 = \partial\partial^\dagger + \partial^\dagger\partial \tag{49}$$

$$\Delta_{\bar{\partial}} = (\bar{\partial} + \bar{\partial}^\dagger)^2 = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial} \tag{50}$$

where $\Delta_\partial : \Omega^{r,s}(M) \longrightarrow \Omega^{r,s}(M)$ is the ∂-Laplacian and $\Delta_{\bar{\partial}} : \Omega^{r,s}(M) \longrightarrow \Omega^{r,s}(M)$ is the $\bar{\partial}$-Laplacian. On a Hermition maifold there exists no relation between the real Δ, the ∂-Laplacian and $\bar{\partial}$-Laplacian.

On a real manifold, a harmonic form is a form ω that satisfies $\Delta\omega = 0$; on a Hermition manifold, a ∂-harmonic form is a bidegree (r, s)-form ω, that satisfies $\Delta_\partial\omega = 0$. Correspondingly, a $\bar{\partial}$-harmonic is a bi-degree (r, s)-form ω, satsfying $\Delta_{\bar{\partial}}\omega = 0$. The following theorem is the complex analogue of the theorem stating that a form is harmonic on a compact real maifold, if and only if the form is both closed and co-closed.

Theorem 6.1.1:

Let ω be a ∂-harmonic ($\bar{\partial}$-harmonic) form, ω then satisfies $\partial\omega = \partial^\dagger\omega = 0$ ($\bar{\partial}\omega = \bar{\partial}^\dagger\omega = 0$).

Proof:

Consider the positive definite condition of the pointwise inner product on a Hermition manifold: $(\beta, \beta) \geq 0$. Now consider $(\omega, \Delta\omega) = (\omega, \partial\partial^\dagger\omega) + (\omega, \partial^\dagger\partial\omega) = (\partial\omega, \partial\omega) + (\partial^\dagger\omega, \partial^\dagger\omega) \geq 0$. If we now stipulate that $\Delta\omega = 0$ the LHS vanishes, but in order for the RHS to vanish both $\partial\omega = 0$ and $\partial^\dagger\omega = 0$. The corresponding case in brackets is similarly proveable. □

It is convention when concerning Hermition or Kähler manifolds, to label the $\bar{\partial}$-Laplacian $\Delta_{\bar{\partial}}$, simply the Laplacian, oftend denoted just Δ. The $\bar{\partial}$-harmonic is often termed the harmonic (r, s)-form. The set of harmonic (r, s)-forms is denoted ${Harm_{\bar{\partial}}}^{r,s}$. That is

$$ {Harm_{\bar{\partial}}}^{r,s} = \{\omega \in \Omega^{r,s}(M) : \Delta_{\bar{\partial}}\omega = 0\}. \tag{51} $$

We have now defined enough mathematical structure, to state the analogue of the Hodge decomposition theorem for a Hermition manifold. The proof can be found in [10].

Theorem 6.1.2:

$$ \Omega^{r,s}(M) = \bar{\partial}\Omega^{r,s-1}(M) \oplus \bar{\partial}^\dagger\Omega^{r,s+1}(M) \oplus {Harm_{\bar{\partial}}}^{r,s}. \tag{52} $$

That is, the space of (r, s) forms decomposes uniquely into 3 orthogonal spaces. A general bidegree (r, s)-form ω is then uniquely expressed as

$$ \omega = \bar{\partial}\alpha + \bar{\partial}^\dagger\beta + \gamma \tag{53} $$

where $\alpha \in \Omega^{r,s-1}(M)$, $\beta \in \Omega^{r,s+1}(M)$ and $\gamma \in {Harm_{\bar{\partial}}}^{r,s}$.

A corollary to the Hodge decomposition of forms, gives one the unique decomposition of any element in ${Z_{\bar{\partial}}}^{r,s}(M)$ or any $\bar{\partial}^{\dagger}$-closed form := ${Z_{\bar{\partial}}}^{\dagger r,s}(M)$ (this is not standard notation but it seems convenient).

$$
{Z_{\bar{\partial}}}^{r,s}(M) = Ker\bar{\partial} = {Harm_{\bar{\partial}}}^{r,s} \oplus \bar{\partial}\Omega^{r,s-1}(M) \tag{54}
$$

$$
{Z_{\bar{\partial}}}^{\dagger r,s}(M) = Ker\bar{\partial}^{\dagger} = {Harm_{\bar{\partial}}}^{r,s} \oplus \bar{\partial}^{\dagger}\Omega^{r,s-1}(M) \tag{55}
$$

Proof:

By acting with $\bar{\partial}$ on both sides of equation (), the $\bar{\partial}$-exact and the $\bar{\partial}$-harmonic term's vanish for reasons already stated, leaving the condition $\bar{\partial}\omega = \bar{\partial}\bar{\partial}^{\dagger}\beta = 0$. Now considering the inner product $0 = (\bar{\partial}\bar{\partial}^{\dagger}\beta, \beta) = (\bar{\partial}^{\dagger}\beta, \bar{\partial}^{\dagger}\beta) \geq 0$, requires $\bar{\partial}^{\dagger}\beta = 0$. Therefore the $\bar{\partial}^{\dagger}$-exact term in the decomposition is vanishing. Any $\bar{\partial}$-closed (r, s) form ω, can thus be written as $\omega = \bar{\partial}\alpha + \gamma$ with $\alpha \in \Omega^{r,s-1}$ and $\gamma \in {Harm_{\bar{\partial}}}^{r,s}$. A similar procedure works for the $\bar{\partial}^{\dagger}$-closed form. □

6.2 Hodge Theory on Kähler Manifolds

Unlike the Hermition manifold, the Kähler manifold with its added structure (existsence of a closed Kähler form), does establish a relationship between the Laplacians. In fact apart from a constat of proportionality, the Laplacians are all equivalent. The proof is complicated and is again found in [10].

Theorem. 6.2: If (M, g) is a Kähler manifold, $\Delta = 2\Delta_{\partial} = 2\Delta_{\bar{\partial}}$.

A corollary of the above theorem is that a form is holomorphic if and only if it is harmonic with repect to the Kähler metric. This is certainly a non trivial property on a Kähler manifold. That is $\bar{\partial}\omega = 0 \Longleftrightarrow \Delta\omega = 0$.

Proof:

First consider direction $\Longrightarrow$. A holomorphic r-form ω satisfies $\bar{\partial}\omega = 0$. Also consider its adjoint $\bar{\partial}^{\dagger}\omega$. By definition, a holomorphic r-form is an (r, 0) form, meaning there is no expansion of anti-holomorphic basis forms $d\bar{z}^{\mu}$. Noting that the $\bar{\partial}^{\dagger}$ operator lowers the anti-holomorphic part of the bi-degree by 1, the holomorphoc form must vanish. However $\bar{\partial}\omega = \bar{\partial}^{\dagger}\omega = 0$ then also implies that $\Delta = 0$. Considering direction $\Longleftarrow$ now, if $\Delta\omega = 0$, then $\Delta_{\bar{\partial}} = 0$ and as was stated previously a $\bar{\partial}$-harmonic must satisfy $\bar{\partial}\omega = 0$. $\square$

We now define a related space of harmonic forms. Let ${Harm_{\mathbb{C}}}^q$ be the set of complex harmonic k forms that satisfy $\Delta_d = 0$. That is ${Harm_{\mathbb{C}}}^k = Ker(\Delta_d : {\Omega_{\mathbb{C}}}^q \longrightarrow {\Omega_{\mathbb{C}}}^k)$. Using (26), a complex harmonic k-form decomposes into a sum of bi-degrees (r, s) with $r + s = q$

$$
{Harm_{\mathbb{C}}}^k = \bigoplus_{r=0}^{q} Harm^{r,s=q-r} \tag{56}
$$

7 Cohomology on Complex Manifolds

One way of classifying different geometries is by finding particular groups of transformations that leave the geometric structures invariant. This is analogous to finding gauge transformations in physical systems, where physical observables group into equivalence classes that are gauge invariant. Correspondingly, cohomology theory is a way of constructing algebraic invariants on a topological space. The coarsest structure on a manifold is of course its topological space, and thus its topological non-triviality can be characterised by cohomology. In fact, physical problems concerning manifold's can be recast directly into topoligcal problems; something evident in String Theory compactifications [11] (see section on Calabi-Yau).

7.1 Dolbeault Cohomology

Possibily the most important cohomology is that of de Rahm, as it is a relatively simple way to measure the topological non-triviality of a manifold. But how does one generalise the familiar de Rahm cohomology to complex manifolds with complex forms? Due to the splitting of the exterior derivative $d = \partial + \bar{\partial}$, we choose to focus on the $\bar{\partial}$ operator in order to utilise the (r, s)-cocyle and (r, s)-coboundary sets that were defined previously. This is just a convention, as the cohomology could just as easily be defined in terms of the ∂ operator. The corresponding cohomology is called the Dolbeault cohomology, and as one would expect, it depends on the complex structure of the manifold.

We first define the **Dolbault complex** as a sequence of linear maps (noting that $\bar{\partial}$ is nilpotent)

$$\Omega^{r,0}(M) \xrightarrow{\bar{\partial}} \Omega^{r,1}(M) \xrightarrow{\bar{\partial}} .... \xrightarrow{\bar{\partial}} \Omega^{r,m}(M) \xrightarrow{\bar{\partial}} 0. \tag{57}$$

Using the Dolbeault complex, we may now define the $(r,s)^{th}\bar{\partial}$-cohomology group denoted $H_{\bar{\partial}}{}^{r,s}(M)$ by

$$H_{\bar{\partial}}{}^{r,s}(M) = \frac{Ker(\bar{\partial} : \Omega^{r,s}(M) \longrightarrow \Omega^{r,s+1}(M))}{Im(\bar{\partial} : \Omega^{r,s-1}(M) \longrightarrow \Omega^{r,s}(M))} = Z_{\bar{\partial}}{}^{r,s}(M)/B_{\bar{\partial}}{}^{r,s}. \tag{58}$$

An element $[\omega] \in H_{\bar{\partial}}{}^{r,s}(M)$ is an equivalence class of bi-degree (r, s)-forms, that have two properties: they are elements of the (r, s)-cocycle and they differ from one another by an element in the (r, s)-coboundary. That is, $[\omega_1] = \{\omega_2 \in \Omega^{r,s}(M) : \omega_2 \in Z_{\bar{\partial}}{}^{r,s}(M), \omega_1 - \omega_2 \in B_{\bar{\partial}}{}^{r,s}\}$. Elements in the same equivalanece class are of course cohomologous to one another.

In fact, this cohomology is essentially the same space as one already introduced. Looking at (54) for the decomposition of the (r, s)-cocycle space ${Z_{\bar{\partial}}}^{r,s}(M)$, one can easily show that every element of ${H_{\bar{\partial}}}^{r,s}(M)$ has a unique harmonic representative. This can be formalised by introducing an operator P, to project an (r, s) form to its unique harmonic representative, as $P : \Omega^{r,s}(M) \longrightarrow {Harm_{\bar{\partial}}}^{r,s}$. Therefore an identification is made between $[\omega] \in {H_{\bar{\partial}}}^{r,s}(M)$ and $P\omega \in {Harm_{\bar{\partial}}}^{r,s}$. This is captured in the following complexified version of Hodges Theorem.

Theorem 7.1.1: On a compact, orientable, complex manifold M, ${H_{\bar{\partial}}}^{r,s}(M)$ is isomorphic to ${Harm_{\bar{\partial}}}^{r,s}$:

$$ {H_{\bar{\partial}}}^{r,s}(M) \cong {Harm_{\bar{\partial}}}^{r,s}. \tag{59} $$

This theorem implies that $dim_{\mathbb{C}} {Harm_{\bar{\partial}}}^{r,s} = dim_{\mathbb{C}} {H_{\bar{\partial}}}^{r,s}(M) =: h^{r,s}$, where $h^{r,s}$ are the hodge numbers of the complex manifold. These Hodge numbers can be arranged in a Hodge diamond (we show this in section 9.3 for the Calabi-Yau 3-fold). The hodge diamond is of finite size for compact manifolds (we are also always assuming conectivity of M) [12]. For $dim_{\mathbb{C}} = m$, there are $(m + 1)^2$ hodge numbers, however they are not all independent. For example, the added structure of the complex manifold such as Kählerity or Calabi-Yau, restricts these numbers greatly as we will see.

7.2 Hodge numbers on Kähler manifolds

For the remainder of this section, we consider the hodge numbers on a Kähler manifold, and their relation to the familiar betti numbers.

Theorem 7.1.2 : The Hodge numbers on a Kähler manifold (M, g) satisfy the following

1. $h^{r,s} = h^{s,r}$

2. $h^{r,s} = h^{m-r,m-s}$

The proofs can be found in [8]. Roughly, 1) follows from the use of complex conjugation, showing that $Harm_{\bar{\partial}}{}^{r,s} \cong Harm_{\bar{\partial}}{}^{s,r}$ with 1) then following immediately from theorem 7.1.1. 2) follows from the Poincare duality between $H_{\bar{\partial}}{}^{r,s}(M)$ and $H_{\bar{\partial}}{}^{m-r,m-s}(M)$.
The consequnce of these relations is that the Hodge diamond acquires a horizontal and vertical symmetry. Depending on whether m is even or odd, the number of independent Hodge numbers reduces respectively to $(\frac{1}{2}m+1)^2$ and $\frac{1}{4}(m+1)(m+3)$.

Using the decomposition theorem for complex harmonic k-forms $Harm_{\mathbb{C}}{}^q = \bigoplus_{r=0}^{q} Harm_{\bar{\partial}}{}^{r,s=q-r}$, and the isomorphism $H_{\bar{\partial}}{}^{r,s}(M) \cong Harm_{\bar{\partial}}{}^{r,s}$, we motivate the following decomposition

$$H^q{}_d(M,\mathbb{C}) = \bigoplus_{r=0}^{q} H_{\bar{\partial}}{}^{r,s=q-r}. \tag{60}$$

Note $H^q{}_d(M,\mathbb{C})$ is not the Dolbeault cohomology but the complexified de Rahm cohomology i.e equivalence classes containing elements of $\Omega^q{}_{\mathbb{C}}(M)$ that are closed with repsect to the d operator, and differ from one another by an exact form $d\alpha$ where $\alpha \in \Omega^{q-1}{}_{\mathbb{C}}(M)$. Therefore it can be seen that the complexified de Rahm cohomology decomposes uniquely into a sum over dolbeault cohomologies.

The Betti numbers are the topological invariants given by the dimension of the de Rahm cohomology: $b^k = dim_{\mathbb{R}} H_d{}^k(M,\mathbb{R}) = dim_{\mathbb{C}} H_d{}^k(M,\mathbb{C})$. This relates the Betti numbers to the complexified cohomology. Now considering the dimension of each side of (60), there exists a relation between the Betti numbers and the hodge numbers (for a compact Kähler manifold only)

$$b^k = \sum_{r=0}^{k} h^{r,k-r}. \tag{61}$$

Two further relations between the Betti numbers and the Hodge numbers can be established on **compact** Kähler manifolds with no boundary ($\partial M = \emptyset$):

1. $b^{2k-1} = 2n, n \in \mathbb{N}$ and $1 \leq p \leq m$

2. $b^{2k} \geq 1, 1 \leq p \leq m$

Relation 1) implies that an odd Hodge number is even. This is because an odd hodge number decomposes only into symmetric pairs which by $b^{r,s} = b^{s,r}$ are equal, meaning the decomposition $b^k = \sum_{r=0}^{k} h^{r,k-r}$ will always factor out a 2. This property can reverse engineer whether a complex manifold admits a Kähler metric; if one of the odd hodge numbers is found to be odd then this automatically excludes it from being Kähler.

Relation 2) : Consider the Kähler from Ω, a top form Ω^m and k-form Ω^k, given by the respective exterior products of Ω. Ω is a closed, real 2-form therefore its k'th wedge product is also a closed real 2k-form. If Ω^k is exact, it can be shown that $VolM \propto \int_M \Omega^m = 0$, using stokes's theorem. But on a compact manifold $VolM > 0$, implying a contradiction. The Ω^k must then not be exact, and so the 2k-form defines a non trivial equivalence class of H^{2k}, with non 0 dimension.

The Euler characteristic $\chi(M)$, is a topological invariant that can be formed entirely from the objects introduced thus far: $\chi(M) = \sum_{r}^{2m} (-1)^r b^r = \sum_{r,s} (-1)^{r+s} h^{r,s}$.

The Euler characteristic is used in many areas of theoretical physics; for example in the calculation of scattering amplitudes in string theory. We will further see (in chapter 9) the remarkable relation between $\chi(M)$ and the number of particle generations in a string compactification [13].

7.3 The Kähler Class

Now that we have introduced both the Kähler metric g, the Kähler form Ω, and the Dolbeault cohomology ${H_{\bar{\partial}}}^{r,s}$, we can define one of the most important equivalence classes in Kähler geometry. Consider the Kähler form Ω on a compact Kähler manifold. Its closure implies it is a representative of an equivalence class of ${H_{\bar{\partial}}}^{1,1}$, i.e $[\omega] \in {H_{\bar{\partial}}}^{1,1}$. We call this particular class the Kähler class.

8 Chern Classes

Chern classes are a type of characteristic classes. That is, they are topological invariants on the vector bundles of a manifold. They simultaneously obstruct the vector bundle from being its trivial bundle $M \times F$, and provide a necessary condition on two isomorphic bundles; namely that they must have the same chern class. As characteristic classes are subsets of the cohomology classes, so too are the Chern classes. These classes appear in various areas of mathematics and physics, and as such have many equivalent definitions.

The following is a brief introduction to Chern classes, employing features of gauge theory and differential geometry that would be familiar to a physicist. We particularly focus on the constructions that will be pertinant to Calabi-Yau theory.

8.1 What are Chern classes?

Consider a complex vector bundle $E \xrightarrow{\pi} M$ of rank r, where the dimension of M is m (see appendix for an introduction to complex vector bundles). Let $\mathcal{F}$ be a 2-form, and $\mathcal{A}$ be a connection 1-form of the bundle with structure Lie group G. The Lie algebra generators are in some representation T_α, satisfying $[T_\alpha, T_\beta] = f_{\alpha\beta}{}^{\gamma} T_\gamma$. From a gauge theory perspective, $\mathcal{F}$ is identified as the Yang-Mill field strength, while $\mathcal{A}$ is identitifed with the gauge potential. We express the field strength in terms of the potnetial as

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}, \tag{62}$$

where the action of $\mathcal{F}$ on vectors of the tangent bundle is $\mathcal{F}(X, Y) = d\mathcal{A}(X, Y) + [\mathcal{A}(X), \mathcal{A}(Y)]$. On a chart with coordinates x^μ, we can write

$$\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu \tag{63}$$

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu, \tag{64}$$

where the components of the field strength are then $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$. Because $\mathcal{F}_{\mu\nu}$ and $\mathcal{A}_\mu$ are $\mathfrak{g}$-valued functions, we can express them in terms of the generators of the Lie algebra as

$$\mathcal{F}_{\mu\nu} = \mathcal{F}_{\mu\nu}{}^{\alpha} T_\alpha \tag{65}$$

$$\mathcal{A}_\mu = \mathcal{A}_\mu{}^{\alpha} T_\alpha, \tag{66}$$

where $\mathcal{F}_{\mu\nu}{}^{\alpha} = \partial_\mu \mathcal{A}_\nu{}^{\alpha} - \partial_\nu \mathcal{A}^{\alpha} + f_{\beta\gamma}{}^{\alpha} \mathcal{A}_\mu{}^{\beta} \mathcal{A}_\nu{}^{\gamma}$. The Yang-Mill equation of motion for the non abelian gauge field (ignoring the gauge coupling constant) is given in terms of the covariant derivative

$$D_{[\rho}\mathcal{F}_{\mu\nu]} = \partial_{[\rho}\mathcal{F}_{\mu\nu]}{}^{\alpha} + f_{\beta\gamma}{}^{\alpha}A^{\beta}{}_{[\rho}\mathcal{F}_{\mu\nu]}{}^{\gamma} = 0. \tag{67}$$

Definition 8.1.1 : The **total Chern class** $c(\mathcal{F})$ of a complex vector bundle is defined

$$c(\mathcal{F}) = det(1 + \frac{i}{2\pi}\mathcal{F}) = 1 + c_1(\mathcal{F}) + c_2(\mathcal{F}) + ... + c_r(\mathcal{F}). \tag{68}$$

Assuming the Lie group G is a matrix Lie group, the algebra generators are matrices and the field strength can be represented by a matrix. It is in this sense we can take the determinant of the above combination (where we note 1 as the identity matrix). On the RHS we have the expansion of the total Chern class into a sum over **Chern classes** where the i'th Chern class is $c_i(\mathcal{F})$. We list the first few:

$$c_0(\mathcal{F}) = [1] \tag{69}$$

$$c_1(\mathcal{F}) = \left[\left(\frac{\imath}{2\pi}\right) Tr\mathcal{F}\right] \tag{70}$$

$$c_2(\mathcal{F}) = \left[\frac{1}{2}\left(\frac{\imath}{2\pi}\right)^2 \left(Tr\mathcal{F} \wedge Tr\mathcal{F} - Tr\left(\mathcal{F} \wedge \mathcal{F}\right)\right)\right] \tag{71}$$

$$\vdots \tag{72}$$

$$c_r(\mathcal{F}) = \left[\left(\frac{\imath}{2\pi}\right)^r det\mathcal{F}\right] \tag{73}$$

To be precise, a Chern class is an equivalence class of Chern forms. That is, a Chern class is given by the cohomology class of one of its representatives, where the representative is a particular Chern form.

Properties:

1. $c_i(\mathcal{F}) = 0$ for $i > r$. That is, the series terminates at the r^{th} chern class given by the determinant of $\mathcal{F}$.

2. $c_i(\mathcal{F}) = 0$ for $2i > m$.

3. The total Chern class is closed as it is a scalar. Since $\mathcal{F}$ is a 2-form, the Chern classes must be cohomology classes of closed 2i-forms, so that $[c_i(\mathcal{F})] \in H^{2i}(M, \mathbb{R})$.

4. Chern classes are independent of the choice of connection. A connection A' would rotate the representative within the same cohomology class

We now take the complex vector bundle to be the holomorphic tangent bundle $(T^{(1,0)}M)$. It can be shown that in this case, $\mathcal{F}$ becomes the curvature 2-form $-\imath\Re$ [7]. The first Chern class of this holomorphic bunlde is then completely determined by the Ricci form $\Re$, and given by

$$c_1(\mathcal{F}) = \left[\frac{1}{2\pi}\Re\right] = \left[\frac{1}{2\pi}\imath\partial\bar{\partial}\log det(g_{\mu\bar{\nu}})\right] := c_1(M) \tag{74}$$

Theorem 8.1.1 : $c_1(M)$ is invariant under a change of the metric: $g \to g + \delta g$.

Proof : Recall the identities $\delta det(g_{\mu\bar{\nu}}) = det(g_{\mu\bar{\nu}})g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$ and $\partial\bar{\partial} = -\frac{1}{2}d(\partial - \bar{\partial})$. Now consider

$$\delta\Re = \imath\partial\bar{\partial}g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}} = d\eta$$

where $\eta = -\frac{1}{2}(\partial - \bar{\partial})g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$ is a one form. As $\delta\Re$ is an exact form, the cohomology class $[\Re + \delta\Re]$ represents only a change in the representative by an exact form. Therefore $[\Re] = [\Re + \delta\Re]$ and $c_1(M) \to c_1(M)$. $\Box$

Finally, it is interesting to note that the Euler characteristic defined in section 7.2 can be written in terms of the m'th Chern class [7]:

$$\chi(M) = \int_M c_m(M). \tag{75}$$

Although this has not been motivated we will see in nect section it will have implications for determining the Hodge numbers of the Calabi-Yau 3-fold. We introduce the Calabi-Yau manifold, and for this, we only require the first Chern class.

9 Calabi-Yau Manifolds

9.1 Why Calabi-Yau?

On the question of what fixed, background, space-time geometry is able to accommodate a superstring theory, the candidates must satisfy certain constraints imposed on them by the supersymmetric string theory itself. First and foremost, superstrings exist in 10 dimensions. In order to reconcile this with our local 4 dimensional surroundings, we propose a compactification of an extra 6 dimensional space; small enough so that at familiar low energies, they are unobservable. We represent this as a product manifold of the local minkowski spacetime with an extra 6 dimensional compact manifold: $M_{10} = \mathcal{M}_4 \times M_6$.

Consider a supergravity theory. The solutions to the supergravity Euler-Lagrange equations, admit a global supersymmetry if there exists a Killing spinner; that is a covariantly conserved spinner with respect to the Levi-Civita conection. For low energy standard model physics, only $N = 1$ global supersymmetries are present. Remarkably, this condition is precicely realised if M_6 is a Calabi-Yau manifold (a Calabi-Yau 3 fold has 6 real dimensions). Specifically, at low energies a Calabi-Yau compactification keeps the $N = 1$ supersymmetry, but spontaneously breaks the superfluous supersymmetries . At high energies, any additional unbroken supersymmetries are welcomed as they would extend physics beyond the standard model. If we then include background gauge fields in addition to gravity, we can incoorporate these by creating a more general structure, namely "Generalised Calabi-Yau geometry" [15].

A Calabi-Yau manifold can then be used to construct a supersymmetric compactification of the Heterotic String [14]. For example, consider Heterotic Type 1 Supergravity in 10 dimensions. This theory possesses 16 local supersymmetries. If we use the Calabi-Yau 3-fold as the background, at low energies and in 4 dimensions, a quarter of the local supersymmetries would spontaneously break. The 4 supersymmetries preserved correspond to an N = 1 supersymmetry (recall in 4 dimensions a spinor has 4 components).

9.2 Calabi-Yau Geometry

A covariantly conserved spinner is a crucial property that the background geometry must accommodate in order to give a realistic standard model supersymmetry. In examining this property, one can reverse engineer the geometry that will end up characterising the Calabi-Yau manifold. We start off by showing that a covariantly conserved spinor ϵ, automatically leads to a restriction on the holonomy [20]. Consider the following manipulation:

$$\nabla\epsilon = 0 \Longrightarrow \nabla_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{8}\omega_\mu{}^{ab}\Gamma_{ab}\epsilon = 0 \tag{76}$$

where the covariant derivative is expressed in terms of the spin connection $\omega_\mu{}^{ab}$, and the antiymmetric product of two connections Γ_{ab}. Antisymmetrising, we can form the commutator $[\nabla_\nu, \nabla_\mu]\epsilon$ where

$$[\nabla_\nu, \nabla_\mu]\epsilon = 0 \Longrightarrow \left(R_{\mu\nu ab}\Gamma^{ab}\right)^\alpha{}_\beta \epsilon^\beta = 0. \tag{77}$$

First note that we have used a vielbein to convert to two flat indices a, b and that $\left(R_{\mu\nu ab}\Gamma^{ab}\right)^\alpha{}_\beta$ is a matrix in spinner space. Suppose we consider a 6 dimensional euclidean space with a 4 component spiner. Such a spinner transforms under $Spin(6) \cong SU(4)$ in the usual way as $\epsilon \rightarrow \epsilon' = exp(\frac{1}{8}\Lambda_{ij}\Gamma^{ij})\epsilon$ where $\Lambda_{ij} = -\Lambda_{ji}$. In order to be a 0 of this equation, the subgroup $\rightarrow SU(\frac{n}{2}) = SU(3) \subset SU(4)$. We will return shortly to examine this holonomy in more detail.

We can also show that the restriction $\nabla\epsilon = 0$ has implications on the curvature. Consider the following manipulations:

$$R_{cdab}\Gamma^{ab}\epsilon = 0 \Longrightarrow \Gamma^c R_{cdab}\Gamma^{ab}\epsilon = 0 \Longrightarrow R_{da}\Gamma^a\epsilon = 0 \tag{78}$$

where we have made use of the following properties:

1. $R_{c[dab]} = 0$
2. $\Gamma^c\Gamma^{ab} = \alpha\Gamma^{cab} + \beta\delta^{c[a}\Gamma^{b]}$. Here the product of Γ^c with the antisymmetric Γ^{ab} gives both a totally anysymmetric term and a term with a kronecker delta. (α, β) are constants that are not important due to the equality to zero.

Ultimately, we find that a necessary condition for $\nabla\epsilon = 0$ is Ricci flatness or $\Re = 0$ (recall in Kähler geometry $Ric_{\mu\bar{\nu}} = \Re_{\mu\bar{\nu}}$). Ricci flatness is then said to be an integrability condition for the equation $\nabla\epsilon = 0$. Note that the converse is not true, as there exist many Ricci flat metris that do not lead to covariantly conserved spinners. Thus the condition is necessary but not sufficient. Using the Ricci flatness let us look at the holonomy in more detail now.

In a Kähler system, consider parallely transporting a holomorphic vector X around an infinitesimal parallelogram of sides ϵ and δ. The new vector has components $X'^{\mu} = X^{\mu} + X^{\nu}R^{\mu}{}_{\nu\kappa\bar{\lambda}}\epsilon^{\kappa}\delta^{\bar{\lambda}}$. The components of the holonomy group are then $h_{\mu}{}^{\nu} = \delta_{\mu}{}^{\nu} + R^{\mu}{}_{\nu\kappa\bar{\lambda}}\epsilon^{\kappa}\delta^{\bar{\lambda}}$. The Lie group U(m) has corresponding Lie algebra $\mathfrak{u}(m)$, that decomposes into a tracless part and a trace part as $\mathfrak{u}(m) = \mathfrak{su}(m) \oplus \mathfrak{u}(1)$. Now using the fact that the geometry is Ricci flat, the trace part of the Lie algebra can be shown to vanish as $R^{\kappa}{}_{\kappa\mu\bar{\nu}}\epsilon^{\mu}\delta^{\bar{\nu}} = 0$ [8] implying only the traceless part remains. The holonomy group then becomes $SU(m)$. In comparing this with the Kähler holonomy, we can see that the Ricci flatness takes $U(M) \rightarrow SU(M)$.

We now look more closely at the implications of a vanishing Ricci form $\Re = 0$. Firstly, Ricci flat of course implies a solution to the Einstein field equations in a vacuum ($T_{\mu\nu} = 0$); this is imperative for a theory respecting General Relativity. In fact, excluding the Joyce manifolds, Calabi-Yau manifolds are the only known compact manifolds to satisfy the Einstein equations [5]. Secondly, the following theorem relates this discussion to the Chern classes of chapter 8.

Theorem 9.2.1 : If a Kähler manifold (M, J, g) admits a Ricci flat metric

h, the first chern class $c_1(M) = 0$.

Proof: Theorem 8.1.1 states the invariance of the first Chern class under a change of metric. Therefore we just have to show that for the Ricci flat matric h, the first chern class is 0. But ofcourse this is trivial as $c_1(M) = \left[\frac{1}{2\pi}\Re\right]$, so that setting the Ricci form to 0 immediately gives a vanishing first Chern class for the Kähler manifold that admits such a metric. $\square$

We can go one step further. Consider the Ricci form $\Re = \imath\partial\bar{\partial}\log det(g_{\mu\bar{\nu}})$. Ricci-flatness implies $\log det(g_{\mu\bar{\nu}}) = f + \bar{f}$, where f and $\bar{f}$ are functions that vanish exclusively under one or the other partial derivative.

If we make a holomorphic coordinate transofrmation $z'^{\mu} = z'^{\mu}(z^{\mu})$ (similarly for conjugate coordinate) the metric components transforms as ${g'}_{\mu\bar{\nu}} = g_{\alpha\bar{\beta}}\frac{\partial z^{\alpha}}{\partial z'^{\mu}}\frac{\partial \bar{z}^{\bar{\beta}}}{\partial \bar{z}'^{\bar{\nu}}}$. The log part of the Ricci form then transforms as $\log[det({g'}_{\mu\bar{\nu}})] = \log[det(g_{\alpha\bar{\beta}})] + \log[det\frac{\partial z^{\alpha}}{\partial z'^{\mu}}] + \log[det\frac{\partial \bar{z}^{\bar{\beta}}}{\partial \bar{z}'^{\bar{\nu}}}]$, where the two additional terms are absorbed into the functions f and $\bar{f}$. This shows that the solution to the Ricci form doesn't change form with holomorphic changes in the coordinates.

Due to this, we are able to choose a particular coordinate system such that $\log det(g_{\mu\bar{\nu}}) = 0$, so that $det(g_{\mu\bar{\nu}}) = 1$. Expressing the metric in terms of the Kähler potential yields

$$det\frac{\partial^2 K}{\partial z^{\mu}\partial \bar{z}^{\bar{\nu}}} = 1. \qquad (79)$$

This is the **Monge-Ampére equation**. It is a complicated non-linear differential equation, where in order to find a Calabi-Yau metric, there must exist a solution K. In fact we can motivate this equation another way. Suppose we look at Calabi-Yau 3-fold, and suppose this comes with a nowhere-vanishing (3, 0) form α which is parallel. That is $\nabla\alpha = 0$ implying both $\bar{\partial}\alpha = \partial\alpha = 0$. α then satisfies the folowing properties:

1. $\alpha \wedge \bar{\alpha} = Vol$
2. $\alpha = \frac{1}{6}\epsilon_{\alpha\beta\gamma} dz^{\alpha} \wedge dz^{\beta} \wedge dz^{\gamma} \hat{f}$, where ϵ is the usual antisymmetric cyclic tensor and $\hat{f} \in O(U_i)$ (a holomorphic function).

As a result of these considerations, we can write

$$\hat{f}\bar{\hat{f}} = \sqrt{det(g_{\alpha\bar{\beta}})} = 1. \tag{80}$$

This once again gives the Monge-Ampére equation (once the potential is inserted) that we got to by considering Ricci flatness. It appeared this time however by postulating the existence of a global nowhere-vanishing holomorphic m-form, given by $\alpha = \frac{1}{6}\epsilon_{\alpha\beta\gamma} dz^{\alpha} \wedge dz^{\beta} \wedge dz^{\gamma}$ when we set $\hat{f}$ to 1. As we will see, this will in fact be an equivalent way of defining a Calabi-Yau manifold.

The critical point is the following: there exists a unique solution to the Monge-Ampére equation for a compact manifold of vanishing first chern class. We already saw from Theorem 9.2.1, that a Kähler manifold admitting a metric with vanishing Ricci form, also forces a vanishing first Chern class; but now this implies the converse is also true. This is summarised in the folowing Calabi conjecture which was initially postulated by Calabi and was later proved by Yau [16].

Calabi Conjecture Theorem :

Given a compact Kähler manifold (M, J, g) with Kähler form Ω and vanishing first chern class $c_1(M) = 0$, there exists a unique Ricci flat Kähler metric g'

with associated Kähler form $\Omega' \in [\Omega]$.

This implies the existence of a unique Ricci flat Kähler metric associated to a Kähler form in each equivalence class of ${H_{\bar{\partial}}}^{1,1}$. That is, the unique existence of a Ricci flat Kähler metric for each Kähler class. Recall there are $h^{1,1}$ Kähler classes, so the number of possible such metrics is $h^{1,1}$ for a given Kähler manifold. We return to this point once we have examined the Hodge numbers of a Calabi-Yau manifold.

Using the mathematical machinery of the previous sections, we are now in a position to define the Calabi-Yau manifold.

Definition 9.2.1 : A Calabi-Yau manifold is a compact Kähler manifold (M, J, g) with

1. Zero Ricci form: $\Re = 0$
2. Zero first Chern class: $c_1(M) = 0$
3. Holonomy group $\subseteq SU(m)$
4. Existence of a global, nowhere-vanishing holomorphic m-form.

We have already highlighted some of the geometric interconnections between these conditions. In particular, it was shown how they relate to supersymmetry restrictions, and how they culminate in the Monge-Ampére equation. We now look more closely at the cohomology casses of the Calabi-Yau manifold, and show the remarkable simplifications of the Hodge numbers.

9.3 Cohomology and Hodge Numbers

As a Calabi-Yau manifold is a special type of Kähler manifold, the complex conjugation and Poincare duality restrictions on the Hodge numbers transfer directly over. However, the extra structure on a Calabi-Yau manifold establishes many more relations between the Hodge humbers. We then show the reduction of the number of independent Hodge numbers on a Calabi-Yau 3-fold, which we recall is a candidate for the 6 dimensional compactification of superstring theory.

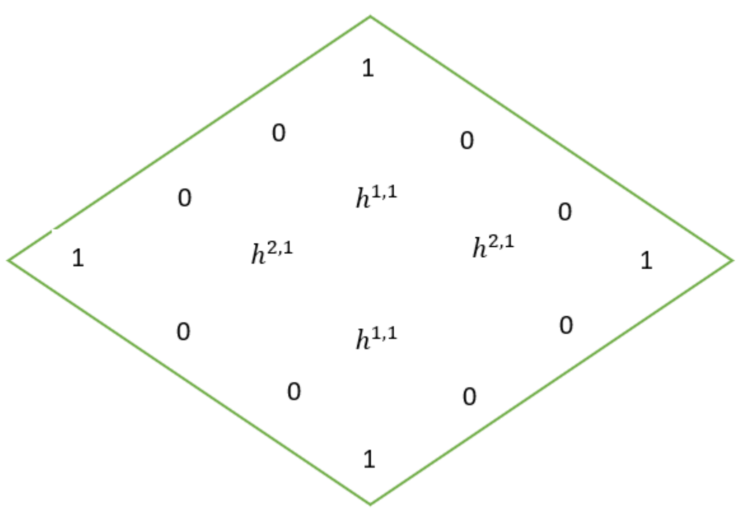
The first simplification we can make to the hodge numbers makes use of the compactness of the Calabi-Yau manifold. Consider $h^{0,0} = dim H_{\bar{\partial}}{}^{0,0}(M) = dim Harm_{\bar{\partial}}{}^{0,0}$, which shows that $h^{0,0}$ is equal to the dimension of the space of harmonic functions. A well known theorem states that on a compact complex manifold, a holomorphic function must be a constant, and thus the space of such functions must be of unit complex dimension: $h^{0,0} = 1$. Using the Hodge duality $h^{r,s} = h^{m-r,m-s}$, we immediately have $h^{m,m} = 1$. This determines the top and bottom of the Hodge diamond.

The next simplification makes use of the existence of a no-where vanishing (m, 0) form. In fact this condition is equivalent to the triviality of the canonical bundle [12] (see apendix for description of the canonical bundle) of the Calabi-Yau manifold: $K_M = \wedge^{m,0} = M \times \mathbb{C}$. Holomorphic sections of the canonical bundle are holomorphic volume forms proportional to $dz^1 \wedge ... \wedge dz^m$, as this is the only basis vector. As the bundle is trivial, we can always pick out one non-vanishing, globally defined (m, 0) form α, which corresponds to the unit section $M \times \{1\}$. Since α is a holomorphic form giving $[\alpha] \in H^{m,0}$, and due to the single basis vector, any other holomorphic volume form will be given by $f\alpha$ where f is some complex function. But in order to remain a holomorphic volume form, the function f must also be holomorphic and thus a constant function. Multiplication by a constant will not change the cohomology class, and so we can instead define the class with the representative $f\alpha$ as $[f\alpha] \in H^{m,0}$. As a result of this, there is only one cohomology class for holomorphic forms and $h^{m,0} = h^{0,m} = 1$.

We now explore another relation called holomorphic duality. Consider the Calabi-Yau 3-fold to be precise. For $[\beta] \in H^{0,s}$ there exists a unique $[\gamma] \in H^{0,3-s}$ such that

$$\int_M \alpha \wedge \beta \wedge \gamma = 1. \tag{81}$$

Note that here we are integrating over a (3, 3) top form and by stokes theorem, integrating over the enitre manifold give unity. This shows that $h^{0,s} = h^{0,3-s}$. The final Hodge condition is that $h^{1,0} = 0$. This comes from $0 = b^1 = 2h^{1,0}$ on a compact Kähler manifold. Using all the relations outlined above, we can now write out the Hodge diamond for the Calabi-Yau 3-fold:



We can now make note of the symmetries of the Hodge diamond (for a

Calabi-Yau 3-fold).

Note first the horizontal and vertical symmetry which came from the Kähler structure. Note also that the sides are composed of 0's and 1's. This is actaully true for Calabi-Yau m-forms in general. We then see that the only undetermined (independent) Hodge numbers are $h^{1,1}$ and $h^{2,1}$. As expected, Calabi-Yau manifolds of higher dimension will have generally more independent Hodge numbers. Note also that $h^{1,1} \geq 1$, as there must exist at least one non trivial Kähler class on a Calabi-Yau manifold for the existence of a Ricci flat metric. In principle, the properties of the particular Calabi-yau manifold would determine these other Hodge numbers. Recalling the formula for the Euler characteristic $\chi(M) = \sum_{r,s}(-1)^{r+s}h^{r,s}$, we can express χ in terms of the unknown Hodge numbers as

$$\chi(M) = 2(h^{1,1} - h^{2,1}). \tag{82}$$

The Euler Characteristic can often be calculated by other means. For example, $\chi(M)$ can be found from $\chi(M) = \int_M c_m(M) = \int_M c_3(M)$ (from the section on Chern classes) in the 3-fold case. Once calulated, the independent Hodge numbers would be reduced to just finding $h^{1,1}$. In fact, we can take this one step further if we consider the following. A remarkable result of certain Calabi-Yau compactifications, is that the number of particle generations ends up being proportional to the Euler characteristic. That is, a topological inavriant of a manifold will characterise the number of generation's of particles it admits. For example, the $E_8 \times E_8$ heterotic string compactification leads to the number of generations being equal to $\frac{1}{2}|\chi|$ [13]. For 3 particle generations as in the standard model, $\chi(M)$ would have to be ± 6, so that

$$\pm 3 = (h^{1,1} - h^{2,1}), \tag{83}$$

which would again eliminate one of the independent Hodge numbers. A rigid Calabi-Yau[] manifold is defined to be one with $h^{2,1} = 0$, and so the Hodge numbers are completely determined in this case as $h^{1,1}$ would have to be 3 (must be greater than 0 so cannot be -3).

Another interesting phenomenon relating the Hodge diamond to theoretical physics is mirror symmetry [17]. This is when $h^{1,1}$ and $h^{2,1}$ are flipped in the Hodge diamond. This new Calabi-Yau manifold, with its own distinct geometry, appears to lead to completely equivalent string theory compactifications. This is a vast topic and is only mentioned as an interesting note of further study. This ends our discussion on Calabi-Yau manifolds and their various geoemtric properties.

10 Generalised Calabi-Yau

Generalised geometry encapsulates a generalisation of many of the mathematical structures we developed thus far. A natural question is why is this needed? Didn't the geometry culminating in the Calabi-Yau manifold, suffice in providing a suitable background for a supersymmetric string theory? Of course the devil is in the details, as the type of string compactification has all to do with answering that question. For example, if we want a supergravity theory with additional fields such as the dilaton ϕ and H field, we would have to generalise the Calabi-Yau geometry in some way to accommodate these.

The methodology behind generalising a Calabi-Yau manifold, consits in generalising each of the following:

1. Complexified tangent budle, $T_{\mathbb{C}}M$

2. Almost complex structure, J

3. Lie bracket, $[,]|_L$

4. Kähler structure and potential, K

5. Calabi-Yau structure and Monge-Ampére equation, $det\frac{\partial^2 K}{\partial z^\mu \partial \bar{z}^{\bar{\nu}}} = 1$

The prescription for generlaising the above structures is far from trivial, and different generalisations are in fact used. [18] prescription is chosen, as it couples more naturally to the physics. This section is mainly for conceptual completeness; as such, we summarise the bottom line equations that are relevant, leaving the details to [18, 2] .

10.1 Generalised Calabi-Yau Geometry

The generalised geometry approach begins with the replacement of the tangent bundle with the tangent bundle plus the cotangent bundle: $TM \to TM \oplus T^*M$. A natural inner product is defined on this space as

$$< X + \omega, Y + \eta >= \frac{1}{2}(i_X \eta + i_Y \omega) \tag{84}$$

where $X, Y \in \Gamma(TM)$ and $\omega, \eta \in \Gamma(T^*M)$ and recalling i_V is the interior derivative with respect to vector field V. Under this inner product, the sub-bundle of $TM \oplus T^*M$ such that all possible inner products of vector fields (sections of the sub bundle) are vanishing is called the isotropic sub-bundle. This sub-bundle is maximally isotropic if after picking a section, you keep adding only other sections that are isotrpic with respect to it until no more can be added.

Definition 10.1.1 :

The Lie bracket is geenralised to a Courant bracket given by

$$[X+\omega, Y+\eta]|_C = \{X,Y\} + \mathcal{L}_X\eta - \mathcal{L}_Y\omega - \frac{1}{2}d(i_X\eta - i_Y\omega) \tag{85}$$

Properties:

1. Diffeomorphically invariant under M just as the Lie bracket was (can say that Courant bracket forms the algebra of a genralised differomorphism [19]).
2. Does not satisfy the Jacobi identity unlike the Lie bracket.

Definition 10.1.2 :

A maximally isotropic sub-bundle with closed sections under the Courant bracket (involutive) is labeled a dirac structure.

Analogous to the complex decomposition of the tangent bundle in chapter 2.2, we have the decomposition of the new generalised bundle:

$$(TM \oplus T^*M) \otimes \mathbb{C} = L + \bar{L} \tag{86}$$

where L is a dirac structure. This can equivalently be achieved by generalising the almost complex structure J to an endomorphism on $TM \oplus T^*M$.

We now begin generalising Kähler geomtry. We introduce two of the above almost complex structure and label them $\mathcal{J}_1$ and $\mathcal{J}_2$ where $[\mathcal{J}_1, \mathcal{J}_2] = 0$. The Gualtieri map [18] relates this to a manifold with a metric and a B field labeled (g, B) with complex structures J_+ and J_-. Connections are defined

$$\nabla^{\pm} : {\Gamma_{ij}}^{\pm k} = {\Gamma_{ij}}^{k} \pm \frac{1}{2}{H_{ij}}^{k} \tag{87}$$

where ${\Gamma_{ij}}^k$ are the ordianry Levi-Civita connection components. These can be shown to give

$$\nabla^+ J_+ = 0 \Longrightarrow Hol(\nabla^+) \subset U(m) \tag{88}$$

$$\nabla^- J_- = 0 \Longrightarrow Hol(\nabla^-) \subset U(m) \tag{89}$$

just as in Kähler geometry. The next step is to relate the differential forms on M to spinors ρ on the generalised bundle such that $(X+\eta)\rho = i_X \rho + \eta \wedge \rho$. Using an invariant bi-linear form between two such spinors ρ_1 and ρ_2, we form the Mukai pairing, denoted (ρ_1, ρ_2). The precise evaluation is given in [2].

We still need some more structure however. We must first constrain ρ_1 and ρ_2 to be closed, pure spinors. Purity implies the spinor annihilates a maximally isotropic sub-bundle of the generalised bundle (or generalised complexified bundle). In this case each spinor is defining a generalised complex structure: $\mathcal{J}_1, \mathcal{J}_2$.

Definition 10.1.3 :

The existence of two closed, pure spinners ρ_1 and ρ_2 that define their own generalised complex structures, and also satisfy

$$(\rho_1, \bar{\rho}_1) = \alpha(\rho_2, \bar{\rho}_2) \neq 0, \tag{90}$$

are said to form the the generalised Calabi-Yau metric structure. Note $\alpha \neq 0$ is a constant.

The normalisation of the spinors is now $(\rho_1, \bar{\rho}_1) = Vol \exp(-2\phi)$. We see the dilaton field ϕ being introduced through the generalisation. Therefore we now have the tripple (g, B, ϕ). Infact the metric and B field are given locally by the generalised Kähler potential. For completeness we state the remaining generalistions and again postpone the details to [2].

1. $\Re \rightarrow {R_{ij}}^{+} + 2{\nabla_i}^{(-)}\nabla_j \phi = 0$ where ${R_{ij}}^{+}$ is the trace of the curvature with respect to the ∇^{+} connection which has torsion. This is the generalisation fo the Ricci flatness.

2. $det \frac{\partial^2 K}{\partial z^\mu \partial \bar{z}^{\bar{\nu}}} = 1 \rightarrow exp(-4\phi)\sqrt{det(g_{\mu\nu}} = 1$. Again we see the presence of a dilaton term that ends up generalising the Monge-Ampére equation.

11 Conclusion

In conclusion, the rich structure of complex geometry has been introduced. The emphasis of the complex geometry in this thesis was on the construction of Calabi-Yau manifolds, whereby starting from a basic almost complex structure and Riemannian metric, we build up towards Calabi-Yau geometry. The Calabi-Yau 3-fold is examined in particular. This manifold was shown to have properties suitable for flux-less compactifications in 10 dimensional superstring theories. A more thorough exposition of this is given in [21]. Considerable time was also spent on forming and analysing the symmetries of its Hodge diamond. Finally a very brief introduction to generalised Calabi-Yau geometry is sketched out. More varied and detailed expositions on this vast subject can be found in [1, 2].

There are still many open questions relating to complex geometry.For example, one can always consider various specifications of the 2 independent Hodge numbers of the Calabi-Yau 3-fold, and thus probe any resulting interesting geometries. The briefly mentioned mirror symmetry, is another popular phenomena specific to Calabi-Yau manifolds, with implications for both pure mathematics and theoretical physics. For example, the relation to T-duality in theoretical physics is examined in [22]. Moreover, testament to the complexity of Calabi-Yau geometry, the Calabi-Yau 3-fold has never even been completely classified and may even be infinite in number[23]. This is also a possible direction one could pursue.

12 Appendix

12.1 Complex Vector Bundles

Vector bundles are fibre bundles whose fibre has a vector space structure. Complex vector bundles are then vector bundles with complex vector spaces, i.e vector spaces over a complex scalar field, such that a general fiber $F = \mathbb{C}^r$.

A complex vector bundle over a complex manifold consists of the following elements:

1. Total space : smooth manifold E where $dim_{\mathbb{C}}E = m + r$.
2. Base space : Complex manifold M where $dim_{\mathbb{C}}M = m$.
3. Fibre : smooth fibre manifold $F = \mathbb{C}^r$ where r = fibre dimension := rank of bundle.
4. Structure group : lie group $G = GL(r, \mathbb{C})$ giving a left action on each fibre.
5. Surjective projection map : $\pi : E \to M$ such that $\pi^{-1}(p) = \mathbb{C}^r|_p \cong \mathbb{C}^r$.
6. Local trivialisation : set of diffeomorphisms $\{\phi_i\} : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}^r$, where $\{U_i\}$ is an open covering of M such that $\pi \circ {\phi_i}^{-1}(p, \mathbb{C}^r|_p) = p$.
7. For $U_i \cap U_j \neq \emptyset$, the transition functions $t_{ij} : \mathbb{C}^r \longrightarrow \mathbb{C}^r$ are elements of G.

An example of a complex vector bundle is $\wedge^{r,s}M$. Its sections are the (r, s)-forms and the space of all such sections is $\Gamma(\wedge^{r,s}M)$. The complex vector bundle is labeled trivial when the transition functions are the identity maps, and the bundle is then given by the direct product $M \times \mathbb{C}^r$. In fact, this is the most trivial example of a structure called the holomorphic vector bundle. A holomorphic vector bundle requires *additional* structure imposed on the complex vector bundle:

1. E must also be a complex manifold.
2. $\pi : E \to M$ is a holomorphic map between the two complex manifolds.
3. $\{\phi_i\}$ local trivialisations are biholomorphic maps. That is ϕ and ϕ^{-1} are both holomorphic maps.
4. The transition functions T_{ij} are holomoprhic maps.

Note the definition of a holomorphic map is given in section 2.1. Examples of non trivial holomorphic vector bundles, include the complexified tangent and cotangent bundles, given by $T_{\mathbb{C}}M$ and ${T_{\mathbb{C}}}^*M$ (section 2.2).

We can also form a holomorphic vector bundle by taking the subset $s = 0$ of the previously defined complex vector bundle $\wedge^{r,s}M$, to form $\wedge^{r,0}M$. Holomorphic sections of this bundle are the holomorphic r-forms which we examine in section 3.3. If we further take the case $r = m$, the resulting vector bundle is called the holomorphic line bundle. Its fibre $F = \mathbb{C}$, so that it is has a bundle rank of 1. Note that the tensor product of holomorphic line bundles are also holomorphic line bundles. Sections of this bundle are termed holomorphic volume forms. We call this holomorphic line bundle the canonical bundle of a complex manifold M, given by

$$K_M = \wedge^{m,0}. \tag{91}$$

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